The World Treasury of

PHYSICS, ASTRONOMY, AND MATHEMATICS

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PART ONE

The Realm of the Atom

As physicists probe nature on scales a billion times smaller than the unaided eye can see, they find much that is strange, to be sure, but also ample evidence of connections linking the tiny precincts of the subatomic world to the wider universe. The electrons orbiting each atomic nucleus obey weird rules — performing quantum leaps, for instance, which means disappearing from one spot and appearing at another without having traversed the space in between — but it is thanks to this odd behavior that atoms can link up in fantastic combinations, generating the chemistry of everything from rocks to living cells. Radioactive atoms decay in ways that are impossible to predict in detail, yet large collections of such atoms decay at rates statistically so reliable that they provide scientists with natural clocks that can be consulted for information about the ages of the earth, sun, and stars.

Science often stands accused of shattering nature's wholeness into bite-sized bits for analysis, but the results of such analysis have a way of resurrecting the whole. The laws of thermodynamics, originally investigated in order to design more efficient steam engines, are currently being applied to the study of black holes. Einstein's special theory of relativity, which he composed in an effort to understand better the infinitesimal particles of which light is composed, turned out to be relevant to such wildly divergent concerns as the longevity of interstellar astronauts (the faster you go, the more slowly you age) to how the stars shine (by releasing a little of the energy locked up inside atoms). Even Heisenberg's uncertainty principle, which revealed that there is a fundamental limit to our ability to predict quantum events, has implications that work more to involve us in nature than to alienate us from it: If, as the uncertainty principle implies, no atom can be observed without disturbing it, then the illusion of the passive observer has been shattered, and we are left with no choice but to recognize ourselves as active participants in the atomic world, as cosmic meddlers who of necessity leave our fingerprints everywhere.

ones. There is the pressure of the nature of mathematics itself — of the elusiveness of truth, of the ever-present necessity for skepticism. And, finally, there is the non-mathematical world, in which the mathematician appears unable to find success, and which at almost all points accords the mathematician a monolithic indifference. So there is no way out for mathematicians; there is no place for them to turn but to other mathematicians and inward on themselves. The insanity and suicide levels among mathematicians are probably the highest in any of the professions. But the rewards are proportionately great. A new mathematical result, entirely new, never before conjectured or understood by anyone, nursed from the first tentative hypothesis through labyryinths of false attempted proofs, wrong approaches, unpromising directions, and months or years of difficult and delicate work — there is nothing, or almost nothing, in the world that can bring a joy and a sense of power and tranquility to equal those of its creator. And a great new mathematical edifice is a triumph that whispers of immortality. What is more, mathematics generates a momentum, so that any significant result points automatically to another new result, or perhaps to two or three other new results. And so it goes — goes, until the momentum all at once dissipates. Then the mathematical career is, essentially, over; the frustrations remain, but the satisfactions have vanished. It has been said that no man should become a philosopher before the age of forty. Perhaps no man should remain a mathematician after the age of forty. The world is, after all, full of worlds to conquer.

BENOIT B. MANDELBROT

We think of geometry as ancient — as the science of Euclid and of the Egyptian surveyors — but it is also a fertile, living discipline. The potential of its undiscovered riches was demonstrated anew in the 1970s with the discovery, by Benoit B. Mandelbrot (b. 1924), of *fractals,* a new field in geometry capable of generating and interpreting structures the complexity and beauty of which rival that of nature's own. Disinclined to underestimate his own abilities (he had taught economics at Harvard, engineering at Yale, and physiology at the Einstein School of Medicine), Mandelbrot proclaimed the news of fractals in a heroic tone that prompted one reviewer of his book *The Fractal Geometry of Nature* to remark approvingly, "I like people who write for glory and not just for money."

How Long Is the Coast of Britain?

Fractal Geometry

Why is geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

More generally, I claim that many patterns of Nature are so irregular and fragmented, that, compared with *Euclid* — a term used in this work to denote all of standard geometry — Nature exhibits not simply a higher degree but an altogether different level of complexity. The number of distinct scales of length of natural patterns is for all practical purposes infinite.

The existence of these patterns challenges us to study those forms that Euclid leaves aside as being "formless," to investigate the morphology of the "amorphous." Mathematicians have disdained this challenge, however, and have increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel.

Responding to this challenge, I conceived and developed a new geometry of nature and implemented its use in a number of diverse fields. It describes many of the irregular and fragmented patterns around us, and leads to full-fledged theories, by identifying a family of shapes I call *fractals.* The most useful fractals involve *chance,* and both their regularities and their irregularities are statistical. Also, the shapes described here tend to be *scaling,* implying that the degree of their irregularity and/or fragmentation is identical at all scales. The concept of *fractal* (Hausdorff) *dimension* plays a central role in this work.

Some fractal sets are curves or surfaces, others are disconnected "dusts," and yet others are so oddly shaped that there are no good terms for them in either the sciences or the arts.

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F. J. Dyson has given an eloquent summary of this theme of mine:

"*Fractal* is a word invented by Mandelbrot to bring together under one heading a large class of objects that have [played] . . . an historical role . . . in the development of pure mathematics. A great revolution of ideas separates the classical mathematics of the 19th century from the modern mathematics of the 20th. Classical mathematics had its roots in the regular geometric structures of Euclid and the continuously evolving dynamics of Newton. Modern mathematics began with Cantor's set theory and Peano's space-filling curve. Historically, the revolution was forced by the discovery of mathematical structures that did not fit the patterns of Euclid and Newton. These new structures were regarded . . . as 'pathological,' . . . as a 'gallery of monsters,' kin to the cubist painting and atonal music that were upsetting established standards of taste in the arts at about the same time. The mathematicians who created the monsters regarded them as important in showing that the world of pure mathematics contains a richness of possibilities going far beyond the simple structures that they saw in Nature. Twentieth-century mathematics flowered in the belief that it had transcended completely the limitations imposed by its natural origins.

"Now, as Mandelbrot points out, . . . Nature has played a joke on the mathematicians. The 19th-century mathematicians may have been lacking in imagination, but Nature was not. The same pathological structures that the mathematicians invented to break loose from 19th-century naturalism turn out to be inherent in familiar objects all around us."[1]

In brief, I have confirmed Blaise Pascal's observation that imagination tires before Nature. . . .

Fractal geometry reveals that some of the most austerely formal chapters of mathematics had a hidden face: a world of pure plastic beauty unsuspected till now. . . .

How Long Is the Coast of Britain?

To introduce a first category of fractals, namely curves whose fractal dimension is greater than 1, consider a stretch of coastline. It is evident that its length is at least equal to the distance measured along a straight line between its beginning and its end. However, the typical coastline is irregular and winding, and there is no question it is much longer than the straight line between its end points.

There are various ways of evaluating its length more accurately, and this chapter analyzes several of them. The result is most peculiar: coastline length turns out to be an elusive notion that slips between the fingers of one who wants to grasp it. All measurement methods ultimately lead to the conclusion that the typical coastline's length is very large and so ill determined that it is best considered infinite. Hence, if one wishes to compare different coastlines from the viewpoint of their "extent," length is an inadequate concept.

This chapter seeks an improved substitute, and in doing so finds it impossible to avoid introducing various forms of the fractal concepts of dimension, measure, and curve.

MULTIPLICITY OF ALTERNATIVE METHODS OF MEASUREMENT

METHOD A: Set dividers to a prescribed opening ε, to be called the yardstick length, and walk these dividers along the coastline, each new step starting where the previous step leaves off. The number of steps multiplied by ε is an approximate length $L(\varepsilon)$. As the dividers'

1. From "Characterizing Irregularity" by Freeman Dyson, *Science,* May 12, 1978, vol. 200, no. 4342, pp. 677–678. Copyright © 1978 by the American Association for the Advancement of Science.

opening becomes smaller and smaller, and as we repeat the operation, we have been taught to expect $L(\epsilon)$ to settle rapidly to a well-defined value called the *true length*. But in fact what we expect does not happen. In the typical case, the observed $L(\varepsilon)$ tends to increase without limit.

The reason for this behavior is obvious: When a bay or peninsula noticed on a map scaled to 1/100,000 is reexamined on a map at 1/10,000, subbays and subpeninsulas become visible. On a 1/1,000 scale map, sub-subbays and sub-subpeninsulas appear, and so forth. Each adds to the measured length.

Our procedure acknowledges that a coastline is too irregular to be measured directly by reading it off in a catalog of lengths of simple geometric curves. Therefore, METHOD A replaces the coastline by a sequence of broken lines made of straight intervals, which are curves we know how to handle.

METHOD B: Such "smoothing out" can also be accomplished in other ways. Imagine a man walking along the coastline, taking the shortest path that stays no farther from the water than the prescribed distance ε. Then he resumes his walk after reducing his yardstick, then again, after another reduction; and so on, until ε reaches, say 50 cm. Man is too big and clumsy to follow any finer detail. One may further argue that this unreachable fine detail (a) is of no direct interest to Man and (b) varies with the seasons and the tides so much that it is altogether meaningless. We take up argument (a) later on in this chapter. In the meantime, we can neutralize argument (b) by restricting our attention to a rocky coastline observed when the tide is low and the waves are negligible. In principle, Man could follow such a curve down to finer details by harnessing a mouse, then an ant, and so forth. Again, as our walker stays increasingly closer to the coastline, the distance to be covered continues to increase with no limit.

METHOD C: An asymmetry between land and water is implied in METHOD B. To avoid it, Cantor suggests, in effect, that one should view the coastline with an out-of-focus camera that transforms every point into a circular blotch of radius ε. In other words, Cantor considers all the points of both land and water for which the distance to the coastline is no more than ε. These points form a kind of sausage or tape of width 2ε. . . . Measure the area of the tape and divide it by 2ε. If the coastline were straight, the tape would be a rectangle, and the above quotient would be the actual length. With actual coast-

lines, we have an estimated length $L(\varepsilon)$. As ε decreases, this estimate increases without limit.

METHOD D: Imagine a map drawn in the manner of pointillist painters using circular blotches of radius ε. Instead of using circles centered on the coastline, as in METHOD C, let us require that the blotches that cover the entire coastline be as few in number as possible. As a result, they may well lie mostly inland near the capes and mostly in the sea near the bays. Such a map's area, divided by 2ε, is an estimate of the length. This estimate also "misbehaves."

ARBITRARINESS OF THE RESULTS OF MEASUREMENT

To summarize the preceding section, the main finding is always the same. As ε is made smaller and smaller, every approximate length tends to increase steadily without bound.

In order to ascertain the meaning of this result, let us perform analogous measurements on a standard curve from Euclid. For an interval of straight line, the approximate measurements are essentially identical and define the length. For a circle, the approximate measurements increase but converge rapidly to a limit. The curves for which a length is thus defined are called *rectifiable*.

An even more interesting contrast is provided by the results of measurement on a coastline that Man has tamed, say the coast at Chelsea as it is today. Since very large features are unaffected by Man, a very large yardstick again yields results that increase as ε decreases.

However, there is an intermediate zone of ε's in which $L(\varepsilon)$ varies little. This zone may go from 20 meters down to 20 centimeters (but do not take these values too strictly). But $L(\varepsilon)$ increases again after ε becomes less than 20 centimeters and measurements become affected by the irregularity of the stones. Thus, if we trace the curves representing $L(\varepsilon)$ as a function of ε, there is little doubt that the length exhibits, in the zone of ε's between $\varepsilon = 20$ meters and $\varepsilon = 20$ centimeters, a flat portion that was not observable before the coast was tamed.

Measurements made in this zone are obviously of great practical use. Since boundaries between different scientific disciplines are largely a matter of conventional division of labor between scientists, one might restrict geography to phenomena above Man's reach, for example, on scales above 20 meters. This restriction would yield a well-defined value of geographical length. The Coast Guard may well choose to use the same ε for untamed coasts, and encyclopedias and almanacs could adopt the corresponding $L(\varepsilon)$.

However, the adoption of the same ε by all the agencies of a government is hard to imagine, and its adoption by all countries is all but inconceivable. For example (Richardson 1961), the lengths of the common frontiers between Spain and Portugal, or Belgium and Netherlands, as reported in these neighbors' encyclopedias, differ by 20%. The discrepancy must in part result from different choices of ε. An empirical finding to be discussed soon shows that it suffices that the ε differ by a factor of 2, and one should not be surprised that a small country (Portugal) measures its borders more accurately than its big neighbor.

The second and more significant reason against deciding on an arbitrary ε is philosophical and scientific. Nature does exist apart from Man, and anyone who gives too much weight to any specific ε and $L(\varepsilon)$ lets the study of Nature be dominated by Man, either through his typical yardstick size or his highly variable technical reach. If coastlines are ever to become an object of scientific inquiry, the uncertainty concerning their lengths cannot be legislated away. In one manner or another, the concept of geographic length is not as inoffensive as it seems. It is not entirely "objective." The observer inevitably intervenes in its definition.

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THE RICHARDSON EFFECT

The variation of the approximate length $L(\varepsilon)$ obtained by METHOD A has been studied empirically in Richardson 1961, a reference that chance (or fate) put in my way. I paid attention because I knew of Lewis Fry Richardson as a great scientist whose originality mixed with eccentricity. . . . We are indebted to him for some of the most profound and most durable ideas regarding the nature of turbulence, notably the notion that turbulence involves a self-similar cascade. He also concerned himself with other difficult problems, such as the nature of armed conflict between states. His experiments were of classic simplicity, but he never hesitated to use refined concepts when he deemed them necessary.

. . . [Richardson concluded] that there are two constants, which we shall call λ and D, such that — to approximate a coastline by a broken line — one needs roughly $F\varepsilon^{-D}$ intervals of length ε, adding up to the length

$$L(\varepsilon) \sim F\varepsilon^{1-D}.$$

The value of the exponent D seems to depend upon the coastline that is chosen, and different pieces of the same coastline, if considered separately, may produce different values of D. To Richardson, the D in question was a simple exponent of no particular significance. However, its value seems to be independent of the method chosen to estimate the length of a coastline. Thus D seems to warrant attention.

A COASTLINE'S FRACTAL DIMENSION

Having unearthed Richardson's work, I proposed that, despite the fact that the exponent D is not an integer, it can and should be interpreted as a dimension, namely, as a fractal dimension. Indeed, I recognized that all the above listed methods of measuring $L(\varepsilon)$ correspond to nonstandard generalized definitions of dimension already used in pure mathematics. The definition of length based on the coastline being covered by the smallest number of blotches of radius ε is used in Pontrjagin & Schnirelman 1932 to define the covering dimension. The definition of length based on the coastline being covered by a tape of width 2ε implements an idea of Cantor and Minkowski, and the corresponding dimension is due to Bouligand. Yet these two examples only hint at the many dimensions (most of them known only to a few specialists) that star in diverse specialized chapters of mathematics. . . .

Why did mathematicians introduce this plethora of distinct definitions? Because in some cases they yield distinct values. Luckily, however, such cases are never encountered in this Essay, and the list of possible alternative dimensions can be reduced to two that I have not yet mentioned. The older and best investigated one dates back to Hausdorff and serves to define fractal dimension; we come to it momentarily. The simpler one is similarity dimension: it is less general, but in many cases is more than adequate. . . .

Clearly, I do not propose to present a mathematical proof that Richardson's D is a dimension. No such proof is conceivable in any natural science. The goal is merely to convince the reader that the notion of length poses a conceptual problem, and that D provides a manageable and convenient answer. Now that fractal dimension is injected into the study of coastlines, even if specific reasons come to be challenged, I think we shall never return to the stage when $D = 1$ was accepted thoughtlessly and naively. He who continues to think that $D = 1$ has to argue his case. . . .

HAUSDORFF FRACTAL DIMENSION

If we accept that various natural coasts are really of infinite length and that the length based on an anthropocentric value of ε gives only a partial idea of reality, how can different coastlines be compared to each other? Since infinity equals four times infinity, every coastline is four times longer than each of its quarters, but this is not a useful conclusion. We need a better way to express the sound idea that the entire curve must have a "measure" that is four times greater than each of its fourths.

A most ingenious method of reaching this goal has been provided by Felix Hausdorff. It is intuitively motivated by the fact that the linear measure of a polygon is calculated by adding its sides' lengths without transforming them in any way. One may say (the reason for doing so will soon become apparent) that these lengths are raised to the power $D = 1$, the Euclidean dimension of a straight line. The surface measure of a closed polygon's interior is similarly calculated by paving it with squares, and adding the squares' sides raised to the power $D = 2$, the Euclidean dimension of a plane. When, on the other hand, the "wrong" power is used, the result gives no specific information: the area of every closed polygon is zero, and the length of its interior is infinite.

Let us proceed likewise for a polygonal approximation of a coastline made up of small intervals of length ε. If their lengths are raised to the power D, we obtain a quantity we may call tentatively an "approximate measure in the dimension D." Since according to Richardson the number of sides is $N = F\varepsilon^{-D}$, said approximate measure takes the value $F\varepsilon^{D}\varepsilon^{-D} = F$.

Thus, *the approximate measure in the dimension* D *is independent of* ε. With actual data, we simply find that this approximate measure varies little with ε.

In addition, the fact that the length of a square is infinite has a simple counterpart and generalization: a coastline's approximate measure evaluated in any dimension d smaller than D tends to ∞ as $\varepsilon \to O$. Similarly, the area and the volume of a straight line are zero. And when d takes any value larger than D, the corresponding ap-

proximate measure of a coastline tends to O as $\varepsilon \to O$. The approximate measure behaves reasonably if and only if $d = D$.

A CURVE'S FRACTAL DIMENSION MAY EXCEED 1; FRACTAL CURVES

By design, the Hausdorff dimension preserves the ordinary dimension's role as exponent in defining a *measure.*

But from another viewpoint, D is very odd indeed: it is a fraction! In particular, it exceeds 1, which is the intuitive dimension of curves and which may be shown rigorously to be their topological dimension D_T.

I propose that curves for which the fractal dimension exceeds the topological dimension 1 be called *fractal curves.* And the present chapter can be summarized by asserting that, within the scales of interest to the geographer, coastlines can be modeled by fractal curves. Coastlines are *fractal patterns.*